BRUNN-MINKOWSKI AND PRÉKOPA-LEINDLER'S INEQUALITIES UNDER PROJECTION ASSUMPTIONS

MARÍA A. HERNÁNDEZ CIFRE AND JESÚS YEPES NICOLÁS

ABSTRACT. Brunn-Minkowski's theorem says that $\mathrm{vol}\big((1-\lambda)K+\lambda L\big)^{1/n}$, for K, L convex bodies, is a concave function in λ, and assuming a common hyperplane projection of K and L, it is known that the volume itself is concave. The 'a priori' natural hypothesis of a common $(n-k)$-plane projection of the sets turned out in the end not to imply the $(1/k)$-th concavity of the volume function. In this paper we show which is the, somehow, best projection type assumption that is needed in order to get concavity for $\mathrm{vol}\big((1-\lambda)K+\lambda L\big)^{1/k}$, characterizing also the equality case in the corresponding inequality. Moreover, we consider the same problem for its functional analogue: the Prékopa-Leindler inequality.

1. INTRODUCTION

Let $\mathcal{K}^n$ be the set of all convex bodies, i.e., nonempty compact convex sets, in the n-dimensional Euclidean space $\mathbb{R}^n$. The n-dimensional volume of a measurable set $M \subset \mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\mathrm{vol}(M)$ (or $\mathrm{vol}_n(M)$ if the distinction of the dimension is useful).

Relating the volume with the Minkowski (vectorial) addition of convex bodies, one is led to the famous Brunn-Minkowski inequality. One form of it states that if $K, L \in \mathcal{K}^n$ and $0 \leq \lambda \leq 1$, then

$$(1.1) \qquad \mathrm{vol}\big((1-\lambda)K + \lambda L\big)^{1/n} \geq (1-\lambda)\mathrm{vol}(K)^{1/n} + \lambda\mathrm{vol}(L)^{1/n},$$

i.e., the n-th root of the volume is a concave function. Equality for some $\lambda \in (0, 1)$ holds if and only if K and L either lie in parallel hyperplanes or are homothetic.

The functional version of the Brunn-Minkowski inequality is known as the Prékopa-Leindler inequality ([13, 19], see also [10, Theorem 8.14]). It states that if $\lambda \in (0, 1)$ and $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ are non-negative measurable functions such that, for any $x, y \in \mathbb{R}^n$,

$$h\big((1-\lambda)x + \lambda y\big) \geq f(x)^{1-\lambda}g(y)^{\lambda},$$

then

$$\int_{\mathbb{R}^n} h(x)\,\mathrm{d}x \geq \left(\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x\right)^{1-\lambda}\left(\int_{\mathbb{R}^n} g(x)\,\mathrm{d}x\right)^{\lambda}.$$

The Brunn-Minkowski inequality is one of the most powerful results in Convex Geometry and beyond: for instance, its previously mentioned equivalent analytic version and the fact that the convexity/compactness assumption can be 'weakened' to consider just Lebesgue measurable sets (see [14]), have allowed it to move in much wider fields. It implies very important inequalities as the isoperimetric and Urysohn inequalities (see e.g. [22, p. 382]) or even the Aleksandrov-Fenchel inequality (see e.g. [22, s. 7.3]), and it has been the starting point for new developments like the so called L_p-Brunn-Minkowski theory (see e.g. [15, 16]), a Brunn-Minkowski result for integer lattices (see [8]), or a reverse Brunn-Minkowski inequality (see e.g. [17]), among many others. It would not be possible to collect here all references regarding versions, applications and/or generalizations on the Brunn-Minkowski inequality. So, for extensive and beautiful surveys on them we refer to [1, 7].

In [3, s. 50], the following linear refinement of the Brunn-Minkowski inequality was obtained for convex bodies having a hyperplane projection of the same measure (see also [9, ss. 1.2.4]). To state this result we need to introduce some notation: the set of all k-dimensional (linear) planes of $\mathbb{R}^n$ will be denoted by $\mathcal{L}_k^n$ (in the same way, for $H \in \mathcal{L}_k^n$, we will write for short $\mathcal{L}_i^n(H)$ to denote the set of all i-dimensional (linear) planes of $\mathbb{R}^n$ which are contained in H). Moreover, if $K \in \mathcal{K}^n$, the orthogonal projection of K onto H will be denoted by $K|H$, and with $H^\perp \in \mathcal{L}_{n-k}^n$ we will represent the orthogonal complement of H.

Theorem A ([3]). *Let $K, L \in \mathcal{K}^n$ be convex bodies such that there exists a hyperplane $H \in \mathcal{L}_{n-1}^n$ with $\mathrm{vol}_{n-1}(K|H) = \mathrm{vol}_{n-1}(L|H)$. Then, for all $\lambda \in [0,1]$,*

$$(1.2) \qquad \mathrm{vol}\big((1-\lambda)K + \lambda L\big) \geq (1-\lambda)\mathrm{vol}(K) + \lambda\mathrm{vol}(L).$$

We observe that the above theorem does not mean that the volume functional is concave. In order to assure the volume concavity one should assume a common hyperplane projection itself (see [22, Note 3 for Section 7.7]).

In [20, Theorem 1.2] it was shown that, under the assumption of Theorem A, (1.2) holds with equality if and only if either K and L lie in parallel hyperplanes or one of them, say K, is a *sausage* with respect to the other body, L, namely,

$$(1.3) \qquad K = \sigma + L, \quad \text{with } \sigma \in \mathcal{K}^n, \dim \sigma \leq 1.$$

There exists a general version of the Brunn-Minkowski inequality for mixed volumes (see e.g. [22, Theorem 7.4.5]) and, in [21], Schneider proved in a very elegant way that even this general one admits a refinement of the above type, unifying different results in the literature about this topic (see also [22, s. 7.7]): if $K, L \in \mathcal{K}^n$ are such that there exists $H \in \mathcal{L}_{n-1}^n$ with

$K|H = L|H$, then any mixed volume (in particular, the volume) of the convex combination $(1-\lambda)K + \lambda L$ is a concave function in $\lambda \in [0,1]$.

At this point it can be natural to wonder whether the concavity of the functional $\mathrm{vol}((1-\lambda)K + \lambda L)^{1/k}$ may be achieved under certain conditions on K and L involving projections onto $(n-k)$-planes, $k \in \{2, \ldots, n-1\}$. In relation to the precedent results, one might think that the appropriate hypothesis would be to consider bodies with a *common projection (or equal measure projection) onto an $(n-k)$-plane*. However, in [11] it was shown that this assumption is not enough:

Counterexample 1.1. *There exist two convex bodies $K, L \in \mathcal{K}^n$, $n \geq 3$, with a common $(n-2)$-dimensional projection $K|H = L|H$, $H \in \mathcal{L}^n_{n-2}$, such that, for all $\lambda \in (0,1)$,*

$$(1.4) \qquad \mathrm{vol}\big((1-\lambda)K + \lambda L\big)^{1/2} < (1-\lambda)\mathrm{vol}(K)^{1/2} + \lambda\mathrm{vol}(L)^{1/2}.$$

For instance, in dimension 3, examples of convex bodies having a common 1-dimensional projection and satisfying (1.4) are $L = B_3$ and $K = M + B_3$ (see Figure 1, left), where $M \in \mathcal{K}^2$ is a planar convex body such that its area and perimeter verify a certain condition (see [11, eq. (11)]); as usual, B_n denotes the n-dimensional Euclidean unit ball. The common projection is obtained onto $H = (\mathrm{lin}\, M)^{\perp}$, being $\mathrm{lin}\, M$ the linear hull of M.
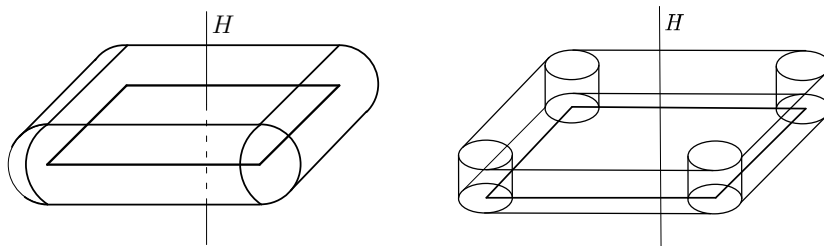

FIGURE 1. The left-hand convex body and B_3 satisfy (1.4); the right-hand set and $B_2 \times [-1,1]$ satisfy (1.5).

It is an easy computation to check that if the ball B_3 is replaced by the archimedean cylinder $C = B_2 \times [-1,1]$ in the above construction (see Figure 1, right), then the expected inequality

$$(1.5) \qquad \mathrm{vol}\big((1-\lambda)K + \lambda L\big)^{1/2} \geq (1-\lambda)\mathrm{vol}(K)^{1/2} + \lambda\mathrm{vol}(L)^{1/2}$$

holds. We observe that the main difference between both examples is a kind of "tomographic discrepancy": for any plane $\bar{H}$ containing H, the set $M + B_3$ (also B_3) has sections $(M+B_3) \cap (u+\bar{H})$, $(M+B_3) \cap (v+\bar{H})$ (for suitable $u, v \in \bar{H}^{\perp}$) having different projections onto H, whereas it does not occur when $L = C$ (with an appropriate choice of $\bar{H}$). In other words, when considering C and $M + C$, the condition on the projection is guaranteed over every section through parallel planes to $\bar{H}$.

In this paper we look for, in the above sense, the *best possible* condition to be assumed in order to obtain the desired inequality. The next theorem provides a solution to this question (see Remark 2.1).

Before stating the result, we need the following additional notation, which will be used throughout the paper: given $K \in \mathcal{K}^n$ and $\bar{H} \in \mathcal{L}^n_{n-m}$, we write

$$K(u) = \big\{ x \in K : x = u + y, \, y \in \bar{H} \big\} = K \cap (u + \bar{H})$$

for any $u \in \bar{H}^\perp$ such that $K \cap (u + \bar{H}) \neq \emptyset$. We observe that $K(u)$ depends on the chosen hyperplane $\bar{H}$; we will use however this notation in order not to make it more involved and because no distinction will be necessary. Moreover, for the sake of brevity, and throughout the paper, the expression "for any $u \in \bar{H}^\perp$" when referring to a section $K(u)$ will mean "for any $u \in \bar{H}^\perp$ such that $K \cap (u + \bar{H}) \neq \emptyset$".

Theorem 1.1. *Let $k \in \{1, \dots, n\}$, $n \geq 3$, and let $K, L \in \mathcal{K}^n$ be convex bodies such that there exist $\bar{H} \in \mathcal{L}^n_{n-k+1}$ and $H \in \mathcal{L}^n_{n-k}(\bar{H})$ satisfying*

$$(1.6) \qquad \mathrm{vol}_{n-k}\big(K(u)|H\big) = \mathrm{vol}_{n-k}\big(L(v)|H\big) \quad \textit{for any } u, v \in \bar{H}^\perp.$$

Then, for all $0 \leq \lambda \leq 1$,

$$(1.7) \qquad \mathrm{vol}\big((1-\lambda)K + \lambda L\big)^{1/k} \geq (1-\lambda)\mathrm{vol}(K)^{1/k} + \lambda\mathrm{vol}(L)^{1/k}.$$

This result will be proved in Section 2, where also the equality case in (1.7) will be characterized (Theorem 2.1). At the beginning of the section, we will briefly discuss about condition (1.6), which is not so restrictive as it might look.

Regarding the Prékopa-Leindler inequality, and following the spirit of the above refinements for the Brunn-Minkowski inequality, the analogue of Theorem A was proved in [6]: under an equal projection assumption for the functions f and g, the Prékopa-Leindler inequality becomes linear in λ. The authors also proved that this linearity can be achieved under the less restrictive hypothesis that the integral of the projections coincide. Moreover, since the Prékopa-Leindler inequality has been generalized using the p-means M_p, which lead to the so-called Borell-Brascamp-Lieb inequalities (see [2], [4]), in [6, Theorem 1.6] the above linearity is obtained in this more general setting. In Section 3 all the definitions and explanations of the involved notions can be found.

Theorem B ([6]). *Let $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be p-concave functions, with $-1/n \leq p \leq \infty$. Let $\lambda \in (0, 1)$ and let $h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be a non-negative measurable function such that*

$$(1.8) \qquad h\big((1-\lambda)x + \lambda y\big) \geq M_p\big(f(x), g(y), \lambda\big)$$

for all $x, y \in \mathbb{R}^n$. If there exists $H \in \mathcal{L}^n_{n-1}$ such that

$$\int_H \mathrm{proj}_H(f)(x)\,\mathrm{d}x = \int_H \mathrm{proj}_H(g)(x)\,\mathrm{d}x,$$

then

$$\int_{\mathbb{R}^n} h(x)\,\mathrm{d}x \geq (1-\lambda)\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x + \lambda\int_{\mathbb{R}^n} g(x)\,\mathrm{d}x.$$

If we consider the characteristic functions $f = \chi_K$ and $g = \chi_L$ of the convex bodies K, L providing Counterexample 1.1, and we take $h = \chi_{(1-\lambda)K+\lambda L}$, then we also get a counterexample to the consequent question whether an inequality of the type

$$\left(\int_{\mathbb{R}^n} h(x)\,\mathrm{d}x\right)^{1/2} \geq (1-\lambda)\left(\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x\right)^{1/2} + \lambda\left(\int_{\mathbb{R}^n} g(x)\,\mathrm{d}x\right)^{1/2}$$

can be obtained if there exists an equal (integral) projection for the functions f and g onto an $(n-2)$-dimensional plane.

In Section 3 we provide the best possible condition to be assumed in order to obtain the desired inequality: again, equality on (the integral of) projections of "sections" f_u, $g_{u'}$ of the functions will be necessary (see Section 3 for the proper definition). We prove the following result, which generalizes Theorem B when $k = 1$ and provides us with a more general setting (cf. Corollary 2.1) in which the $(1/k)$-powered Brunn-Minkowski inequality holds. To this respect we also point out that, for the sake of simplicity, here we present this result for p-concave functions, although it could be set in a wider context. The reason is that in the proof we apply Theorem B but, nevertheless, a more general version of it for arbitrary measurable f and g is also true (see [6, Theorem 4.3]), provided that two mild (but technical) measurability assumptions hold. As usual in the literature, $|\cdot|_1$ will denote the 1-norm, i.e., $|f|_1 = \int_{\mathbb{R}^n} |f(u)|\,\mathrm{d}u$, for $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ measurable.

Theorem 1.2. *Let $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be p-concave functions, $-1/n \leq p \leq \infty$, with $|f|_1, |g|_1 \neq 0$. Let $\lambda \in (0,1)$ and let $h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be a non-negative measurable function satisfying (1.8) for all $x, y \in \mathbb{R}^n$. Let $k \in \{1, \dots, n\}$ and assume that there exist $\bar{H} \in \mathcal{L}^n_{n-k+1}$ and $H \in \mathcal{L}^n_{n-k}(\bar{H})$ such that*

$$\int_H \mathrm{proj}_H(f_u)(x)\,\mathrm{d}x = \int_H \mathrm{proj}_H(g_v)(x)\,\mathrm{d}x$$

for any $u, v \in \bar{H}^\perp$. Then

$$\left(\int_{\mathbb{R}^n} h(x)\,\mathrm{d}x\right)^{1/k} \geq (1-\lambda)\left(\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x\right)^{1/k} + \lambda\left(\int_{\mathbb{R}^n} g(x)\,\mathrm{d}x\right)^{1/k}.$$

2. The $(1/k)$-powered Brunn-Minkowski inequality

Along this paper, we denote by e_i the i-th canonical unit vector, and we write $[x, y]$ for the (closed) segment with end-points $x, y \in \mathbb{R}^n$.

Before the proof of the theorem, we would like to observe that condition (1.6) in Theorem 1.1 is not too restrictive, in the sense that it implies no "similarity" in the geometry of the sets. For instance, the following convex bodies in $\mathbb{R}^3$ satisfy (1.6) for $\bar{H} = \{x_2 = 0\}$ and $H = \{x_2 = x_3 = 0\}$ (see

Figure 2): the pyramid K defined as the convex hull $K = \mathrm{conv}\{R, p\}$, where $R = [0, a\mathrm{e}_1] + [0, b\mathrm{e}_2] \subset \{x_3 = 0\}$, $a, b > 0$, and $p \in \mathbb{R}^3 \backslash \{x_3 = 0\}$ satisfies that $p|\{x_3 = 0\} \in R$; and the cylinder $L = [0, \mu\mathrm{e}_2] + (b/2)B_2$, $\mu > 0$, where the circle $B_2 \subset \bar{H}$.
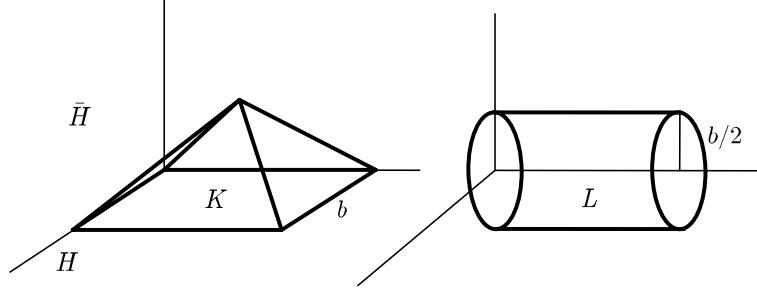

FIGURE 2. Two convex bodies satisfying (1.6).

Roughly speaking, and using the convexity of the sets, (1.6) means that each body (K and L) should have the property that all its nonempty sections by planes parallel to a fixed one have equal projections. Thus, only these projections must be constant over each convex body (and when considering different sets they may differ from each other; only their measures should coincide).

We also notice that, although Theorem 1.1 can be obtained as a consequence of Theorem 1.2 just taking f, g, h as the characteristic functions of the suitable convex bodies, we need a different proof for Theorem 1.1 in order to be able to characterize the equality case.

Throughout the paper, for $K \in \mathcal{K}^n$ and $\bar{H} \in \mathcal{L}^n_{n-k+1}$, we denote by $\varphi_K : \bar{H}^\perp \longrightarrow \mathbb{R}_{\geq 0}$ the function given by

$$\varphi_K(u) = \mathrm{vol}_{n-k+1}\big(K(u)\big), \quad u \in \bar{H}^\perp,$$

and we define the set

$$K_k = \big\{(u, t) \in \mathbb{R}^k : u \in K|\bar{H}^\perp, 0 \leq t \leq \varphi_K(u)\big\}.$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. From (1.6) we also get that

$$\mathrm{vol}_{n-k}\big(K(u)|H\big) = \mathrm{vol}_{n-k}\big(K(v)|H\big) \quad \text{for any } u, v \in \bar{H}^\perp.$$

Therefore, Theorem A yields the linear inequality (1.2) for the sets $K(u)$ and $K(v)$ (for any $u, v \in \bar{H}^\perp$), which implies that

$$\begin{aligned}
\varphi_K\big((1-\lambda)u + \lambda v\big) &= \mathrm{vol}_{n-k+1}\Big(K\big((1-\lambda)u + \lambda v\big)\Big) \\
&\geq \mathrm{vol}_{n-k+1}\big((1-\lambda)K(u) + \lambda K(v)\big) \\
&\geq (1-\lambda)\mathrm{vol}_{n-k+1}\big(K(u)\big) + \lambda\mathrm{vol}_{n-k+1}\big(K(v)\big) \\
&= (1-\lambda)\varphi_K(u) + \lambda\varphi_K(v)
\end{aligned}$$

(and analogously for φ_L). Therefore, φ_K and φ_L are concave functions, which shows that $K_k, L_k \in \mathcal{K}^k$ are convex bodies. Moreover, since

$$(2.1) \qquad \big[(1-\lambda)K + \lambda L\big](u) = \bigcup_{(1-\lambda)u_1 + \lambda u_2 = u} \Big[(1-\lambda)K(u_1) + \lambda L(u_2)\Big],$$

we get, together again with (1.2) now for the sets $K(u)$ and $L(v)$, that

$$(2.2)$$
$$\begin{aligned}
\varphi_{(1-\lambda)K + \lambda L}\big((1-\lambda)u + \lambda v\big) &\geq \mathrm{vol}_{n-k+1}\big((1-\lambda)K(u) + \lambda L(v)\big) \\
&\geq (1-\lambda)\mathrm{vol}_{n-k+1}\big(K(u)\big) + \lambda\mathrm{vol}_{n-k+1}\big(L(v)\big) \\
&= (1-\lambda)\varphi_K(u) + \lambda\varphi_L(v),
\end{aligned}$$

and thus we obtain the inclusion

$$(2.3) \qquad \big[(1-\lambda)K + \lambda L\big]_k \supset (1-\lambda)K_k + \lambda L_k.$$

Finally, using Fubini's theorem, (2.3) and the Brunn-Minkowski inequality in $\mathbb{R}^k$, we can conclude that

$$\begin{aligned}
\mathrm{vol}\big((1-\lambda)K + \lambda L\big)^{1/k} &= \mathrm{vol}_k\Big(\big[(1-\lambda)K + \lambda L\big]_k\Big)^{1/k} \\
&\geq \mathrm{vol}_k\big((1-\lambda)K_k + \lambda L_k\big)^{1/k} \\
&\geq (1-\lambda)\mathrm{vol}_k(K_k)^{1/k} + \lambda\mathrm{vol}_k(L_k)^{1/k} \\
&= (1-\lambda)\mathrm{vol}(K)^{1/k} + \lambda\mathrm{vol}(L)^{1/k},
\end{aligned}$$

which proves (1.7). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We notice that Theorem 1.1 generalizes both, the Brunn-Minkowski inequality (1.1) and Theorem A. Indeed, if $k = n$ then $H = \{0\}$, and hence, with the usual convention that $\mathrm{vol}_0\big(\{0\}\big) = 1$, condition (1.6) trivially holds and thus we have (1.1); for $k = 1$ we get $\bar{H} = \mathbb{R}^n$, which yields $K(u) = K$, and so (1.6) becomes the assumption $K|H = L|H$ in Theorem A.

Remark 2.1. *We observe that condition (1.6) cannot be weakened in the sense of increasing the gap between the dimensions of H and $\bar{H}$, for instance, assuming that $\bar{H} \in \mathcal{L}^n_{n-k+2}$ and $H \in \mathcal{L}^n_{n-k}(\bar{H})$. Indeed, let K and L be the convex bodies providing Counterexample 1.1, and let $H \in \mathcal{L}^n_{n-2}$ be the $(n-2)$-plane such that $K|H = L|H$. Then $\bar{H} = \mathbb{R}^n$, and condition (1.6) turns into $K|H = L|H$. Therefore (1.7) (for $k = 2$) does not hold, but (1.4).*

From the proof of Theorem 1.1 and taking into account that Bonnesen's theorem (Theorem A) also holds in the more general setting of compact sets (see [18]), it can be easily shown that Theorem 1.1 also holds true for compact sets (not necessarily convex):

Corollary 2.1. *Let $k \in \{1, \ldots, n\}$, $n \geq 3$, and let $A, B \subset \mathbb{R}^n$ be nonempty compact sets such that there exist $\bar{H} \in \mathcal{L}^n_{n-k+1}$ and $H \in \mathcal{L}^n_{n-k}(\bar{H})$ satisfying*

$$\mathrm{vol}_{n-k}\big(A(u)|H\big) = \mathrm{vol}_{n-k}\big(B(v)|H\big) \quad \text{for any } u, v \in \bar{H}^{\perp}.$$

Then, for all $0 \le \lambda \le 1$,

$$\mathrm{vol}\big((1-\lambda)A + \lambda B\big)^{1/k} \ge (1-\lambda)\mathrm{vol}(A)^{1/k} + \lambda\mathrm{vol}(B)^{1/k}.$$

We would like to point out that we have set the statement of Theorem 1.1 in the setting of convexity because it is needed to characterize the equality case (see Theorem 2.1 below): for instance, the result collected in [20, Theorem 1.2], which is strongly used along the proof of Theorem 2.1, relies on convexity. We also recall that there is no possible characterization for the equality in the Brunn-Minkowski inequality without convexity.

Next we deal with the equality case in Theorem 1.1. In order to do it, we need some additional notation. Let ϑ_K be the sup-norm of φ_K, i.e.,

$$\vartheta_K = |\varphi_K|_\infty = \sup\nolimits_u \varphi_K(u).$$

Without loss of generality, when $\dim K = \dim L = n$, we will assume from now on that the sets K, L of our theorem satisfy $\vartheta_K \ge \vartheta_L > 0$. Then, denoting by

$$\vartheta = \mathrm{vol}_{n-k}\big(K(u)|H\big) = \mathrm{vol}_{n-k}\big(L(v)|H\big) \ne 0$$

(for any $u, v \in \bar{H}^\perp$, cf. (1.6)), we define $r : L|\bar{H}^\perp \longrightarrow \mathbb{R}_{\ge 0}$ as

$$r(u) = \frac{\left(\frac{\vartheta_K}{\vartheta_L} - 1\right)\mathrm{vol}_{n-k+1}\big(L(u)\big)}{\vartheta}.$$

Finally, let $\sigma \subset \bar{H}$ be a line segment of length 1, which is orthogonal to H and has its midpoint in H.

Now we state the characterization of the equality when $k < n$; the case $k = n$ is just the equality case in the classical Brunn-Minkowski inequality. Roughly speaking, the result says that in order to have equality in the $(1/k)$-powered Brunn-Minkowski inequality, the sections of K must be sausages of sections of L, where the heights of the corresponding sections are related via the constant ϑ_K/ϑ_L; besides, L (the set having ϑ_L minimal) has to satisfy a particular condition regarding its own sections.

Theorem 2.1. *Let $k \in \{1, \dots, n-1\}$. Under the assumptions of Theorem 1.1, equality holds in (1.7) for some $0 < \lambda < 1$ if and only if K and L either lie in parallel hyperplanes or verify the following two conditions (if $\dim K = \dim L = n$):*

   i) *L satisfies that, for any $u \in L|\bar{H}^\perp$ and all $u_1, u_2 \in L|\bar{H}^\perp$ such that $(1-\lambda)(\vartheta_K/\vartheta_L)u_1 + \lambda u_2 = \big((1-\lambda)(\vartheta_K/\vartheta_L) + \lambda\big)u$,*

$$(1-\lambda)\big(r(u)\sigma + L(u)\big) + \lambda L(u) \supset (1-\lambda)\big(r(u_1)\sigma + L(u_1)\big) + \lambda L(u_2)$$

    (up to translations).

   ii) *$K|\bar{H}^\perp = (\vartheta_K/\vartheta_L)L|\bar{H}^\perp$ and, for all $u \in L|\bar{H}^\perp$,*

$$K\left(\frac{\vartheta_K}{\vartheta_L}u\right) = r(u)\sigma + L(u)$$

    (both, up to translations).

Proof. If K and L lie in parallel hyperplanes, then also $(1 - \lambda)K + \lambda L$ lies in a hyperplane, and hence equality trivially holds in (1.7).

Now we suppose that, say, $\dim L < n$, which yields $\dim L_k < k$. If equality holds in (1.7) then, in particular, there is equality in the classical (k-dimensional) Brunn-Minkowski inequality (1.1) for K_k, L_k. Hence, since $\dim L_k < k$, also $\dim K_k < k$ and K_k, L_k lie in parallel $(k-1)$-planes (we notice that the case $\dim L_k = 0$ and $\dim K_k = k$ is excluded because of (1.6)). Therefore, $\dim K < n$ too, and thus $\mathrm{vol}\big((1 - \lambda)K + \lambda L\big) = 0$. This shows that $(1 - \lambda)K + \lambda L$ lies in a hyperplane parallel to the one containing L, and therefore also K.

Thus, from now on we assume that both $\dim K = \dim L = n$. From the proof of Theorem 1.1 we get that equality holds in (1.7) for some $0 < \lambda < 1$ if and only if there is equality (a) in (2.3) and (b) in the classical Brunn-Minkowski inequality (1.1) for K_k, L_k, i.e., if and only if

(a) $\big[(1 - \lambda)K + \lambda L\big]_k = (1 - \lambda)K_k + \lambda L_k$ and
(b) K_k, L_k are homothetic.

First, we assume (a) and (b). In order to avoid repetition, in the following, all sets equalities will be "up to translations". Since $K_k = cL_k$, we get from the definition of K_k, L_k that $K|\bar{H}^\perp = cL|\bar{H}^\perp$ and $\varphi_K(cu) = c\varphi_L(u)$ for all $u \in L|\bar{H}^\perp$; thus, in particular, $\vartheta_K = |\varphi_K|_\infty = c|\varphi_L|_\infty = c\,\vartheta_L$. Therefore, $c = \vartheta_K/\vartheta_L$ and hence

$$(2.4) \qquad\qquad K_k = \frac{\vartheta_K}{\vartheta_L}L_k.$$

Moreover,

$$(2.5) \qquad K|\bar{H}^\perp = \frac{\vartheta_K}{\vartheta_L}L|\bar{H}^\perp \quad \text{and}$$

$$\varphi_K\left(\frac{\vartheta_K}{\vartheta_L}u\right) = \frac{\vartheta_K}{\vartheta_L}\varphi_L(u) \quad \text{for all } u \in L|\bar{H}^\perp.$$

For the sake of brevity, from now on we will write $\alpha = \vartheta_K/\vartheta_L$. Then, (a) and (2.4) yields

$$\big[(1 - \lambda)K + \lambda L\big]_k = \big((1 - \lambda)\alpha + \lambda\big)L_k,$$

which, together with (2.5), implies that
(2.6)
$$\varphi_{(1-\lambda)K+\lambda L}\Big(\big[(1-\lambda)\alpha+\lambda\big]u\Big) = \big[(1-\lambda)\alpha+\lambda\big]\varphi_L(u) = (1-\lambda)\varphi_K(\alpha u)+\lambda\varphi_L(u)$$

for all $u \in L|\bar{H}^\perp$. Then (cf. (2.2) for αu and u) we get, in particular,

$$(2.7) \quad \begin{aligned} \mathrm{vol}_{n-k+1}&\big((1 - \lambda)K(\alpha u) + \lambda L(u)\big) \\ &= (1 - \lambda)\mathrm{vol}_{n-k+1}\big(K(\alpha u)\big) + \lambda\mathrm{vol}_{n-k+1}\big(L(u)\big), \end{aligned}$$

i.e., equality holds in the linear Brunn-Minkowski inequality (Theorem A, see (1.6)), and therefore $K(\alpha u)$ is a sausage of $L(u)$ in $u + \bar{H}$ (cf. (1.3)).

More precisely, $K(\alpha u) = \ell\sigma + L(u)$, $\ell \geq 0$. Now, using (2.5), since

$$\alpha\,\mathrm{vol}_{n-k+1}\big(L(u)\big) = \alpha\varphi_L(u) = \varphi_K(\alpha u) = \mathrm{vol}_{n-k+1}\big(K(\alpha u)\big)$$
$$= \mathrm{vol}_{n-k+1}\big(\ell\sigma + L(u)\big)$$
$$= \mathrm{vol}_{n-k+1}\big(L(u)\big) + \ell\mathrm{vol}_{n-k}\big(L(u)|H\big)$$
$$= \mathrm{vol}_{n-k+1}\big(L(u)\big) + \ell\vartheta,$$

we get

$$\ell = \frac{(\alpha - 1)\mathrm{vol}_{n-k+1}\big(L(u)\big)}{\vartheta} = r(u),$$

which yields

$$(2.8) \qquad\qquad K(\alpha u) = r(u)\sigma + L(u).$$

Thus, (2.8) and the first condition in (2.5) show ii).

In order to prove i) we observe that

$$(2.9) \qquad \big[(1-\lambda)K + \lambda L\big]\Big(\big[(1-\lambda)\alpha + \lambda\big]u\Big) \supset (1-\lambda)K(\alpha u) + \lambda L(u)$$

(cf. (2.1)). But moreover, both sets in the above inclusion have the same $(n-k+1)$-volume; indeed, since $K(\alpha u)$ is a sausage of $L(u)$ (cf. (2.8)), then the volume of the convex combination $(1-\lambda)K(\alpha u) + \lambda L(u)$ is linear (see (2.7)), and thus, together with (2.6), we have the identity

$$\mathrm{vol}_{n-k+1}\Big(\big[(1-\lambda)K + \lambda L\big]\big(\big[(1-\lambda)\alpha + \lambda\big]u\big)\Big)$$
$$= \varphi_{(1-\lambda)K+\lambda L}\Big(\big[(1-\lambda)\alpha + \lambda\big]u\Big) = (1-\lambda)\varphi_K(\alpha u) + \lambda\varphi_L(u)$$
$$= \mathrm{vol}_{n-k+1}\big((1-\lambda)K(\alpha u) + \lambda L(u)\big).$$

Therefore we have equality in (2.9), and using (2.1) we obtain

$$(1-\lambda)K(\alpha u) + \lambda L(u) \supset (1-\lambda)K(\alpha u_1) + \lambda L(u_2)$$

whenever $(1-\lambda)\alpha u_1 + \lambda u_2 = \big[(1-\lambda)\alpha + \lambda\big]u$. It shows i) because (2.8) holds.

Finally we assume i) and ii), and we have to prove (a) and (b). From ii) we directly get that $K|\bar{H}^\perp = \alpha L|\bar{H}^\perp$ and the sausage-property $K(\alpha u) = r(u)\sigma + L(u)$, which yields

$$(2.10) \qquad
\begin{aligned}
\varphi_K(\alpha u) &= \mathrm{vol}_{n-k+1}\big(r(u)\sigma + L(u)\big) \\
&= \mathrm{vol}_{n-k+1}\big(L(u)\big) + r(u)\mathrm{vol}_{n-k}\big(L(u)|H\big) \\
&= \mathrm{vol}_{n-k+1}\big(L(u)\big) + r(u)\vartheta \\
&= \mathrm{vol}_{n-k+1}\big(L(u)\big) + (\alpha - 1)\mathrm{vol}_{n-k+1}\big(L(u)\big) = \alpha\varphi_L(u).
\end{aligned}$$

Therefore, $K_k = \alpha L_k$, which shows (b). In order to prove (a) we first notice that, since $K_k = \alpha L_k$ and we always have the inclusion (2.3), it suffices to show that

$$(2.11) \qquad\qquad \big[(1-\lambda)K + \lambda L\big]_k \subset \big((1-\lambda)\alpha + \lambda\big)L_k.$$

Now we observe that conditions (i) and (ii) imply, for each $u \in L|\bar{H}^{\perp}$, the inclusion

$$(1-\lambda)K(\alpha u) + \lambda L(u) \supset (1-\lambda)K(\alpha u_1) + \lambda L(u_2),$$

if $u_1, u_2 \in L|\bar{H}^{\perp}$ are such that $(1-\lambda)(\alpha u_1) + \lambda u_2 = ((1-\lambda)\alpha + \lambda)u$. This ensures, in particular, (see (2.1)) that the section of $(1-\lambda)K + \lambda L$ corresponding to the vector $((1-\lambda)\alpha + \lambda)u$ is given precisely by the convex combination

$$\big[(1-\lambda)K + \lambda L\big]\Big(\big[(1-\lambda)\alpha + \lambda\big]u\Big) = (1-\lambda)K(\alpha u) + \lambda L(u).$$

Hence, since $K(\alpha u)$ is a sausage of $L(u)$, the volume of $(1-\lambda)K(\alpha u) + \lambda L(u)$ is linear (cf. (2.7)), and together with (2.10) we get, for any $u \in L|\bar{H}^{\perp}$,

$$\begin{aligned}
\varphi_{(1-\lambda)K+\lambda L}\Big(\big[(1-\lambda)\alpha + \lambda\big]u\Big) &= \mathrm{vol}_{n-k+1}\big((1-\lambda)K(\alpha u) + \lambda L(u)\big) \\
&= (1-\lambda)\mathrm{vol}_{n-k+1}\big(K(\alpha u)\big) + \lambda \mathrm{vol}_{n-k+1}\big(L(u)\big) \\
&= (1-\lambda)\varphi_K(\alpha u) + \lambda \varphi_L(u) \\
&= \big((1-\lambda)\alpha + \lambda\big)\varphi_L(u).
\end{aligned}$$

Finally, since $\big((1-\lambda)\alpha + \lambda\big)u \in \big((1-\lambda)\alpha + \lambda\big)L|\bar{H}^{\perp} = (1-\lambda)K|\bar{H}^{\perp} + \lambda L|\bar{H}^{\perp}$, the above identity shows (2.11) and, therefore, (a). $\qquad \square$

Remark 2.2. *We observe, on the one hand, that condition i) of Theorem 2.1 is not too restrictive, i.e., it does not imply that L is just a convex body with "constant sections". This can be shown by taking a concave function $\xi : [a, b] \subset \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$ and*

$$L = \bigcup_{t \in [a,b]} \Big(\big(C + \xi(t)[-v/2, v/2]\big) \times \{t\}\Big),$$

where $C \subset H$ is the $(n-2)$-dimensional cube of edge-length 1 and $v \in \bar{H} \cap H^{\perp}$, $|v| = 1$. It is easy to check that L satisfies condition i).

On the other hand, one could think that the previously mentioned assumption is redundant and that it could be obtained from the convexity of the body L. It is easy to check that this condition does not hold in the following 3-dimensional example: we take the ball $B_2 \subset \{x_3 = 1\}$ and the orthogonal box $D \subset \{x_3 = 0\}$ with edge-lengths $\pi/2, 2$; then it suffices to consider $L = \mathrm{conv}\{B_2, D\}$, any K such that $\vartheta_K / \vartheta_L = 2$, $\lambda = 1/2$, $u_1 = 1$, $u_2 = 0$ and $u = 2/3$.

2.1. Minkowski's first inequality under additional projections assumptions.

The well-known Minkowski first inequality (see e.g. [22, Theorem 7.2.1]) states that for convex bodies $K, E \in \mathcal{K}^n$,

$$(2.12) \qquad \mathrm{V}\big(K[n-1], E\big)^n \geq \mathrm{vol}(K)^{n-1}\mathrm{vol}(E),$$

where $\mathrm{V}\big(K[n-1], E\big)$ stands for the mixed volume of $n-1$ times K and E. For a deep study of mixed volumes we refer to [22, s. 5.1]. Here, we deal with the corresponding refinement of the above inequality, when working

with additional projections assumptions. To this aim, and for the sake of brevity, we will write

$$S(K; E) = nV\big(K[n-1], E\big),$$

following the standard notation for the surface area.

Theorem 2.2. *Let* $k \in \{1, \ldots, n\}$, $n \geq 3$, *and let* $K, L \in \mathcal{K}^n$ *be convex bodies such that there exist* $\bar{H} \in \mathcal{L}^n_{n-k+1}$ *and* $H \in \mathcal{L}^n_{n-k}(\bar{H})$ *satisfying*

$$\mathrm{vol}_{n-k}\big(K(u)|H\big) = \mathrm{vol}_{n-k}\big(L(v)|H\big) \quad \text{for any } u, v \in \bar{H}^\perp.$$

Then,

$$(2.13) \qquad S(K; E) \geq \mathrm{vol}(K)^{(k-1)/k}\Big[(n-k)\mathrm{vol}(K)^{1/k} + k\mathrm{vol}(E)^{1/k}\Big].$$

We notice that inequality (2.13) is indeed stronger than (2.12) because, by the arithmetic-geometric mean inequality (see e.g. [10, Corollary 1.2]), we have $(n-k)\mathrm{vol}(K) + k\mathrm{vol}(E)^{1/k}\mathrm{vol}(K)^{(k-1)/k} \geq n\mathrm{vol}(K)^{(n-1)/n}\mathrm{vol}(E)^{1/n}$.

Proof. Using the so-called (relative) Steiner formula (see e.g. [10, (14)]) it is easy to check that

$$S(K; E) = \lim_{\lambda \to 0^+} \frac{\mathrm{vol}\big((1-\lambda)K + \lambda E\big) - \mathrm{vol}\big((1-\lambda)K\big)}{\lambda}.$$

Then, applying Theorem 1.1, we have

$$S(K; E) \geq \lim_{\lambda \to 0^+} \frac{\Big[(1-\lambda)\mathrm{vol}(K)^{1/k} + \lambda\mathrm{vol}(E)^{1/k}\Big]^k - (1-\lambda)^n\mathrm{vol}(K)}{\lambda}$$

$$= k\mathrm{vol}(K)^{(k-1)/k}\left(\mathrm{vol}(E)^{1/k} - \mathrm{vol}(K)^{1/k}\right) + n\mathrm{vol}(K)$$

$$= (n-k)\mathrm{vol}(K) + k\mathrm{vol}(E)^{1/k}\mathrm{vol}(K)^{(k-1)/k},$$

which gives the result. $\qquad\qquad\square$

3. ON REFINEMENTS OF THE PRÉKOPA-LEINDLER INEQUALITY

In this section we present a proof of Theorem 1.2. In order to completely clarify the statement of this result and to prove it, we need to introduce further notation and definitions. We start by recalling the notion of p-mean of two non-negative numbers, $p \in \mathbb{R} \cup \{\pm\infty\}$, for which we follow [4].

Let $a, b > 0$ and let $\lambda \in [0, 1]$. If $p \in \mathbb{R}$ and $p \neq 0$, then we set

$$M_p(a, b, \lambda) = \big((1-\lambda)a^p + \lambda b^p\big)^{1/p},$$

and for the cases $p \in \{0, \pm\infty\}$ we define

$$M_0(a, b, \lambda) = a^{1-\lambda}b^\lambda$$

and

$$M_\infty(a, b, \lambda) = \max\{a, b\}, \quad M_{-\infty}(a, b, \lambda) = \min\{a, b\}.$$

Finally, if $ab = 0$, we will define $M_p(a, b, \lambda) = 0$ for all $p \in \mathbb{R} \cup \{\pm\infty\}$. Furthermore, if $p \neq 0$, we allow a, b to be ∞, and in that case, $M_p(a, b, \lambda)$ will be, as usual, the value which is obtained "by continuity".

Regarding the functions that are the main object of study in Theorem 1.2, we give the following definition.

Definition 3.1. *A non-negative function $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ is said to be p-concave, for $p \in \mathbb{R} \cup \{\pm\infty\}$, if*

$$f\big((1 - \lambda)x_1 + \lambda x_2\big) \geq M_p\big(f(x_1), f(x_2), \lambda\big)$$

for all $x_1, x_2 \in \mathbb{R}^n$ and all $\lambda \in (0, 1)$.

A 0-concave function is usually called *log-concave*.

We finally need the notions of "projection" and "section" of a function.

Definition 3.2 (see e.g. [12]). *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ and let $H \in \mathcal{L}_{n-1}^n$ with normal unit vector ν. The* projection *of f onto H is the (extended) function $\mathrm{proj}_H(f) : H \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ defined by*

$$\mathrm{proj}_H(f)(x) = \sup_{\alpha \in \mathbb{R}} f(x + \alpha\nu).$$

The geometric meaning of this definition is easy: the (strict) hypograph of $\mathrm{proj}_H(f)$ is the projection of the (strict) hypograph of f onto H. In particular, the projection of the characteristic function of a set is just the characteristic function of the projection of the set.

Definition 3.3. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ and let $H \in \mathcal{L}_m^n$, that we identify with $\mathbb{R}^m$. For any $u \in H^{\perp}$, the* section *of f through $u + H$ is the function $f_u : H \longrightarrow \mathbb{R}_{\geq 0}$ defined by $f_u(x) = f(u, x)$.*

We observe that for the definition of the function $\mathrm{proj}_H(f_u)$, $u \in \bar{H}^{\perp}$, appearing in the hypothesis of the Theorem 1.2, namely, $\mathrm{proj}_H(f_u)(x) = \sup_{\alpha \in \mathbb{R}} f_u(x + \alpha\nu)$, the vector ν is the only normal unit vector (up to the sign) to H in $\bar{H}$.

We are now in a position to prove the result.

Proof of Theorem 1.2. Without loss of generality and for the sake of brevity, throughout the proof any m-plane $H \in \mathcal{L}_m^n$ will be identified with the coordinate plane $\{x_{m+1} = \cdots = x_n = 0\}$.

We are going to prove a slightly more general result, from which Theorem 1.2 will be obtained as the particular case $m = k$:

> **Claim:** Under the hypothesis of Theorem 1.2, for any $m \in \{1, \dots, k\}$ and any $u, v \in \mathbb{R}^{k-m}$ such that $|f_u|_1, |g_v|_1 \neq 0$,

$$(3.1) \quad \left(\int_{\mathbb{R}^{n-k+m}} h_{(1-\lambda)u+\lambda v}(x)\,\mathrm{d}x\right)^{1/m} \geq (1 - \lambda)\left(\int_{\mathbb{R}^{n-k+m}} f_u(x)\,\mathrm{d}x\right)^{1/m}$$
$$+ \lambda\left(\int_{\mathbb{R}^{n-k+m}} g_v(x)\,\mathrm{d}x\right)^{1/m}.$$

We will show (3.1) by (finite) induction on m. The case $m = 1$ is just Theorem B for the functions f_u and f_v. So, we will suppose that the claim (3.1) is true for $m \in \{1, \dots, k - 1\}$.

Since $m \leq k - 1$, for any $u, v \in \mathbb{R}^{k-m}$ we may write $u = (\bar{u}, t_u)$ and $v = (\bar{v}, t_v)$ for some $\bar{u}, \bar{v} \in \mathbb{R}^{k-m-1}$ and $t_u, t_v \in \mathbb{R}$. Given $\bar{u}, \bar{v} \in \mathbb{R}^{k-m-1}$ such that $|f_{\bar{u}}|_1, |g_{\bar{v}}|_1 \neq 0$, let $F, G, H : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$ be the non-negative functions given by

$$F(t) = \int_{\mathbb{R}^{n-k+m}} f_{(\bar{u},t)}(x)\,\mathrm{d}x, \quad G(t) = \int_{\mathbb{R}^{n-k+m}} g_{(\bar{v},t)}(x)\,\mathrm{d}x,$$

$$H(t) = \int_{\mathbb{R}^{n-k+m}} h_{((1-\lambda)\bar{u}+\lambda\bar{v},t)}(x)\,\mathrm{d}x,$$

and we write

$$C_m = \left[(1 - \lambda)\,|F|_\infty^{1/m} + \lambda\,|G|_\infty^{1/m}\right]^m \quad \text{and} \quad \theta_m = \frac{\lambda\,|G|_\infty^{1/m}}{C_m^{1/m}} \in (0, 1).$$

By induction hypothesis, we have that, for any $t, t' \in \mathbb{R}$ with $F(t), G(t') \neq 0$,

$$H\big((1 - \lambda)t + \lambda t'\big) \geq \left[(1 - \lambda)F(t)^{1/m} + \lambda G(t')^{1/m}\right]^m$$

$$= C_m \left[(1 - \theta_m)\left(\frac{F(t)}{|F|_\infty}\right)^{1/m} + \theta_m\left(\frac{G(t')}{|G|_\infty}\right)^{1/m}\right]^m$$

$$\geq C_m \min\left\{\frac{F(t)}{|F|_\infty}, \frac{G(t')}{|G|_\infty}\right\},$$

and hence we get

$$(3.2) \qquad \frac{H\big((1 - \lambda)t + \lambda t'\big)}{C_m} \geq \min\left\{\frac{F(t)}{|F|_\infty}, \frac{G(t')}{|G|_\infty}\right\}.$$

We notice also that

$$\sup_{t \in \mathbb{R}} \frac{F(t)}{|F|_\infty} = \sup_{t \in \mathbb{R}} \frac{G(t)}{|G|_\infty} = 1,$$

which yields

$$(3.3) \qquad \left\{t \in \mathbb{R} : \frac{F(t)}{|F|_\infty} \geq s\right\}, \left\{t \in \mathbb{R} : \frac{G(t)}{|G|_\infty} \geq s\right\} \neq \emptyset$$

for all $0 < s < 1$. Then, (3.2) together with (3.3) imply that

$$\left\{t \in \mathbb{R} : \frac{H}{C_m}(t) \geq s\right\} \supset (1 - \lambda)\left\{t \in \mathbb{R} : \frac{F}{|F|_\infty}(t) \geq s\right\}$$

$$+ \lambda\left\{t \in \mathbb{R} : \frac{G}{|G|_\infty}(t) \geq s\right\}$$

for all $0 < s < 1$, and thus, the Brunn-Minkowski inequality (in $\mathbb{R}$) leads to

$$\mathrm{vol}_1\left(\left\{t \in \mathbb{R} : \frac{H}{C_m}(t) \geq s\right\}\right) \geq (1-\lambda)\,\mathrm{vol}_1\left(\left\{t \in \mathbb{R} : \frac{F}{|F|_\infty}(t) \geq s\right\}\right)$$
$$+ \lambda\,\mathrm{vol}_1\left(\left\{t \in \mathbb{R} : \frac{G}{|G|_\infty}(t) \geq s\right\}\right).$$

Therefore, from the above inequality, and using Fubini's theorem, we get

$$\int_{\mathbb{R}^{n-k+m+1}} h_{(1-\lambda)\bar{u}+\lambda\bar{v}}(x)\,\mathrm{d}x = C_m \int_{\mathbb{R}} \frac{H}{C_m}(t)\,\mathrm{d}t$$
$$= C_m \int_0^{+\infty} \mathrm{vol}_1\left(\left\{t \in \mathbb{R} : \frac{H}{C_m}(t) \geq s\right\}\right)\,\mathrm{d}s$$
$$\geq C_m \int_0^1 \mathrm{vol}_1\left(\left\{t \in \mathbb{R} : \frac{H}{C_m}(t) \geq s\right\}\right)\,\mathrm{d}s$$
$$\geq C_m \int_0^1 \left[(1-\lambda)\,\mathrm{vol}_1\left(\left\{t \in \mathbb{R} : \frac{F}{|F|_\infty}(t) \geq s\right\}\right)\right.$$
$$\left. + \lambda\,\mathrm{vol}_1\left(\left\{t \in \mathbb{R} : \frac{G}{|G|_\infty}(t) \geq s\right\}\right)\right]\,\mathrm{d}s$$
$$= C_m\left((1-\lambda)\int_{\mathbb{R}} \frac{F}{|F|_\infty}(t)\,\mathrm{d}t + \lambda\int_{\mathbb{R}} \frac{G}{|G|_\infty}(t)\,\mathrm{d}t\right)$$
$$= C_m\left((1-\lambda)\frac{\int_{\mathbb{R}^{n-k+m+1}} f_{\bar{u}}(x)\,\mathrm{d}x}{|F|_\infty} + \lambda\frac{\int_{\mathbb{R}^{n-k+m+1}} g_{\bar{v}}(x)\,\mathrm{d}x}{|G|_\infty}\right).$$

Finally, the claim (3.1) (for $m+1$) is obtained by applying the (reverse) Hölder inequality (see e.g. [5, Theorem 1, p. 178]) with parameter $-1/m$ to the latter expression. $\square$

We observe that Theorem 1.2 remains true if the assumption $|f|_1, |g|_1 \neq 0$ is replaced by the weaker condition $|f_t|_1, |g_{t'}|_1 \neq 0$ for some $t, t' \in \mathbb{R}$, which is the key point in the final step of the induction process.

REFERENCES

[1] F. Barthe, Autour de l'inégalité de Brunn-Minkowski, *Ann. Fac. Sci. Toulouse Math. Ser. 6* **12** (2) (2003), 127-178.

[2] C. Borell, Convex set functions in d-space, *Period. Math. Hungar.* **6** (1975), 111–136.

[3] T. Bonnesen, W. Fenchel, *Theorie der konvexen Körper.* Springer, Berlin, 1934, 1974. English translation: *Theory of convex bodies.* Edited by L. Boron, C. Christenson and B. Smith. BCS Associates, Moscow, ID, 1987.

[4] H. J. Brascamp, E. H. Lieb, On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions and with an application to the diffusion equation, *J. Functional Analysis* **22** (4) (1976), 366–389.

[5] P. S. Bullen, *Handbook of means and their inequalities.* Mathematics and its Applications **560**, Revised from the 1988 original, Kluwer Academic Publishers Group, Dordrecht, 2003.

[6] A. Colesanti, E. Saorín, J. Yepes Nicolás, On a linear refinement of the Prékopa-Leindler inequality, *Canad. J. Math.* **68** (2016), 762–783.

[7] R. J. Gardner, The Brunn-Minkowski inequality, *Bull. Amer. Math. Soc.* **39** (3) (2002), 355-405.

[8] R. J. Gardner, P. Gronchi, A Brunn-Minkowski inequality for the integer lattice, *Trans. Amer. Math. Soc.* **353** (10) (2001), 3995-4024.

[9] A. Giannopoulos, *Convex Geometric Analysis*, Private notes 2009 (unpublished manuscript). `http://www.math.uoc.gr/~apostolo/notes.ps`.

[10] P. M. Gruber, *Convex and Discrete Geometry.* Springer, Berlin Heidelberg, 2007.

[11] M. A. Hernández Cifre, J. Yepes Nicolás, Refinements of the Brunn-Minkowski inequality, *J. Convex Anal.* **21** (3) (2014), 1–17.

[12] B. Klartag, V. D. Milman, Geometry of log-concave functions and measures, *Geom. Dedicata* **112** (2005), 169–182.

[13] L. Leindler, On certain converse of Hölder's inequality II, *Acta Math. Sci. (Szeged)* **33** (1972), 217–223.

[14] L. A. Lusternik, Die Brunn-Minkowskische Ungleichung für beliebige messbare Mengen, *C. R. (Dokl.) Acad. Sci. URSS* **8** (1935), 55–58.

[15] E. Lutwak, The Brunn-Minkowski-Firey theory, I, *J. Differential Geom.* **38** (1) (1993), 131-150.

[16] E. Lutwak, The Brunn-Minkowski-Firey theory, II, *Adv. Math.* **118** (2) (1996), 244-294.

[17] V. D. Milman, Inégalité de Brunn-Minkowski inverse et applications à la théorie locale des espaces normés (An inverse form of the Brunn-Minkowski inequality, with applications to the local theory of normed spaces), *C. R. Acad. Sci. Paris Ser. I Math.* **302** (1) (1986), 25-28.

[18] D. Ohmann, Über den Brunn-Minkowskischen Satz, *Comment. Math. Helv.* **29** (1955), 215–222.

[19] A. Prékopa, Logarithmic concave measures with application to stochastic programming, *Acta Sci. Math. (Szeged)* **32** (1971), 301–315.

[20] E. Saorín Gómez, J. Yepes Nicolás, Linearity of the volume. Looking for a characterization of sausages, *J. Math. Anal. Appl.* **421** (2) (2015), 1081–1100.

[21] R. Schneider, Gemischte Volumina in Kanalscharen, *Geom. Dedicata* **30** (2) (1989), 223-234.

[22] R. Schneider, Convex bodies: The Brunn-Minkowski theory. Cambridge University Press, Cambridge, 2nd expanded edition, 2014.

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE MURCIA, CAMPUS DE ESPINAR-DO, 30100-MURCIA, SPAIN
  E-mail address: `mhcifre@um.es`

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE LEÓN, CAMPUS DE VEGAZANA, 24071-LEÓN, SPAIN
  E-mail address: `jyepn@unileon.es`